Principles of the Design of D-Neuronal Networks II. Composing Simple Melodies *

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Abstract. Digital computer simulation of the prediction trees introduced in Part I of this series is performed. We define the simplest prediction tree arrangements for the analysis and composition of diatonic single-lined melodies. A measure of surprise is defined as the weighted sum of the outputs of the tree surprise-evaluating units. The melody composing process is then considered as an optimization task, the goal of which is to find the tone sequences elicited in trees of as great a mean level of surprise as possible.

1 Introduction: melody compositional process as an optimization task

Psychologists often divide human behaviour into several goal-oriented behaviour modes corresponding to the individual biological motives (sometimes called needs, instincts, drives, propensities). It is supposed that at least some of them are accompanied by specific affections. [1] The matter of our interest will be the study of some mechanisms of the acoustic information processing which regulate (initiate and stop) the "human propensity" called by McDougall [2] "exploration of strange places and things". According to him, the activity performed in the frame of this propensity is accompanied by an emotional variable named "wonder". We shall follow the Meyer's idea [3] that a melody belongs to the artificially created "strange things" which elicit in human beings extreme feelings of wonder (or surprise) because of play with expectations of incoming acoustic events. Strictly definable relationships (e.g. the level of consonancy) between individual tones offer to a compositor extraordinary powerful instruments to perform this play.

Our aim to work out outlines of a neural theory for the composing pitch sequences of a melody, addresses two questions: (i) what groupings of pitch features are extracted by the brain during the processing of a single-lined melody and (ii) how does the brain compute the predictions of later pitch events from those that occured earlier. We shall answer these two questions from the viewpoint of the formalism presented in Part I of this series.

The arrangement of predictions trees for simple melody processing (i.e. the definition of the number of trees, their inputs, starting and shifting tones and other tree parameters) determines the concrete manner of the extraction and prediction of pitch groupings. From all the possible prediction tree arrangements for the processing of simple melodies we choose those for which an effective pitch compositional procedure can be performed, i.e. for which we are able to find the tone sequence with defined metrorhythmics eliciting in the trees the highest possible level of surprise.

In Sect. 2 we present a heuristic optimization algorithm which makes it possible to find the tone sequences eliciting in the defined trees a sufficiently great mean level of surprise. In individual subsections of Sect. 3 we define the arrangements of prediction trees for the analysis and composition of pitch sequences and bring the results of the digital computer simulation. In conclusion (Sect. 4.), a discussion of the results presented in the whole series is performed. In what follows, all references to Sections and formulae of the Part I will be denoted as I.x.y. and (I.x.y), respectively.

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2 Principles of composing melodies by means of prediction trees

There will be two prediction trees (see I.2.4.) processing the same input tone sequence. In what follows these trees will analyse the so-called pitch motions (see Definition 1) which occur in the processed tone sequence at one or at two metrorhythmical hierarchy levels.

Definition 1

A pitch sequence $\mathbf{P^u}(n) = \mathbf{P}(1), \mathbf{P}(2), \ldots, \mathbf{P}(n)$ created from pitches $\mathbf{p}(i)$ of the tone sequence $\mathbf{M}(T) = m(1).\mathbf{p}(1), m(2).\mathbf{p}(2), \ldots, m(T).\mathbf{p}(T)$ will be called the *pitch motion* of $\mathbf{M}(T)$ at the metrorhythmical hierarchy level u if we first choose from $\mathbf{M}(T)$ in unchanged order all pitches $\mathbf{p}(i)$ for which it holds $m(i) \geq u$ and then we omit each pitch which is the same as that directly preceding.

It means that the both prediction trees may be started and shifted only at time-moments when a tone arrives at the tree input the pitch of which is different of that of the previous shifting tone. The difference between both prediction trees consists in the additional requirements put on the definition of the tree starting and shifting tones. Let the first tree is started by each occurence of a tone with a strong accent, i.e. $m(i) = 3$, (and a new pitch) and shifted by an arbitrary accented tone, i.e. $m(i) \geq 2$, (with a new pitch), while the second tree is started by each occurence of an accented tone, i.e. $m(i) \geq 2$ (with a new pitch) and shifted by each onset of a tone, i.e. $m(i) \geq 1$ (with a new pitch). Consider further that both prediction trees contain pathes of the length not more then d, i.e. at the most d D-units are serially connected by c-connections in each tree path. This number is called the *prediction depth* of a prediction tree. However, it must be emphasized that a tree path of the length d contains $d+1$ SE-units (one root SE-unit and d tree node SE-units). The group of all the tree nodes connected with the tree root through k c-conection will be called the kth *level* of the prediction tree.

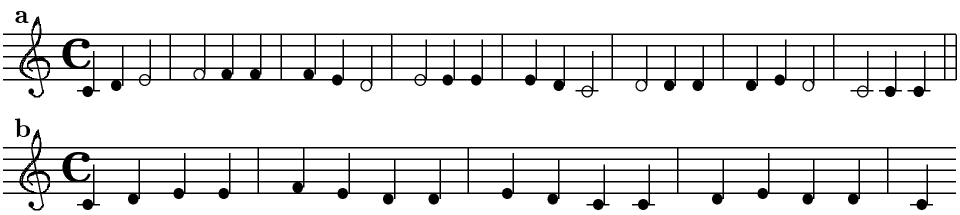

Fig. 1. *A melody and its pitch motion reduction. (a) The Slovak folk song which was used for the prediction tree education. (b) The shortest tone sequence with the 4/4-metrorhythmics containing the same pitch motions as this folk song.*

If $d = 3$ and the both prediction trees process as input the melody depicted in Fig. 1 then the first tree will process subsequently the pitch subsequences c,e,f,d; f,d,e,c; e,c,d,c; d,c; c; of the input melody while the second prediction tree the subsequences c,d,e,f; e,f,e,d; f,e,d,e; f,e,d,e; etc. The continuations of these subsequences simply "fall out" from the corresponding tree paths. In general, since a processed tone sequence should have an emotive effect only if its continuation is surprising at any time in regard to the expectations evaluated for each time by prediction trees, the pitch subsequences of a melody ought to activate in the trees so many surprise-evaluating units and so highly as possible. If we take into account the fact that each tree surprise-evaluating unit together with a D-unit controlled by it, represents in an immature tree a filter "transmitting through this tree node" only the tones which are "similar" to the D-unit permanent inner state and to the current SE-unit memory state, each tone of a melody has at least two different functions: (i) to evoke surprise, and (ii) to open gates in prediction trees, i.e. to "tune" the memory states of SE-units in such a manner that following tones may "pass" through the corresponding tree-nodes and also evoke surprise. However, it is clear that the chords may fulfil such a "SE-unit tuning role" more powerful than single tones.

Let us suppose that the mean level of surprise elicited in the prediction trees by a melody of the length T is given by the formula

$$S = \frac{1}{T} \sum_{t=1}^{T} \sum_{i=1}^{2} \sum_{j=0}^{d} \sum_{k=1}^{n_{ij}} f_i(t).w_i.L_j.S^i_{jk}(t), \tag{1}$$

where: $S^i_{jk}(t)$ is the output of the kth SE-unit of the jth tree level in the ith prediction tree at time t; d is the prediction depth of the trees, n_{ij} is the number of nodes in the jth level of the ith tree, $f_i(t)$ represents the coeficient weighting the "smoothness" of the sequence progression in the ith tree at time t; w_i is the prediction importance of the ith tree in regard to other trees; L_j expresses the prediction importance of the tree-level j in regard to other tree-levels.

In order to find the appropriate factors f, w, L we shall shortly discuss some relevant issues concerning brain mechanisms of the extraction of musical and visual patterns.

Many years before the so-called regularization theories of contour extractions [4] appeared, Meyer [3] stated: "...the perception of a line or motion initiates a mental process, and it is this mental process which, following the mental line of least resistance, tends to be perpetuated and continued." The extraction of a melody line from a sequence of chords may be considered as an analogy to the extraction of contours from a digitalized noise corrupted image. Both tasks represent an ill-posed problem the regularization of which requires to define additional constraints put on possible solutions. Deutch [5] discussing organizational processes in music such as Principles of Proximity, Similarity, and Good Continuation has pointed out that a sequence which changes smoothly in frequency is likely to be coming from a single source and therefore tends to be perceived together. It means that a candidate for a constraint put on extracted melody lines is the line smoothness. The optimalization procedures [6] for image contour extraction also use this constraint.

If we consider the prediction trees as a sort of a short-term memory for prediction and surprise evaluation then a pitch compositional process may be considered being the recall of the tone sequence with a defined metrorhythmics (e.g. 31213121...) from this memory performed by means of the following optimization procedure: find the sequence that is so smooth as possible, and activates in the most important prediction trees so many SE-units so highly, and at so distant tree levels as it is possible.

Thus, the weigthing functions f, w, L from (1) can be defined as follows:

$$f_i(t) = \frac{1}{D_i(t)}, \tag{1a}$$

where $D_i(t) \quad (D_i(t) \in \{1, 2, \ldots, 7\})$ is the difference between the pitch ordinal number of the tone arriving at the ith tree input at time $t \quad (t = 1, 2, \ldots)$ and that of the previous tree shifting tone. The ordinal numbers n of pitches c,d,...,c' are $1, 2, ..., 8$, respectively. We set $n(0) = 0$, i.e. $D(1) = n(1)$. If we want to support the recalling process at one hierarchical level of a melody against its another level we must correspondingly set up the values of prediction importances w_i of the prediction trees so that

$$w_i = k, \tag{1b}$$

where k is an positive integer evaluating the relative importance of the ith tree. Whereas

$$L_j = j + 1, \tag{1c}$$

where $j \quad (j \in \{1, 2, ..., n\})$ is the ordinal number of the tree-level. The meaning of this weighting factor is obvious. The pragmatical value of predictions and surprises is supposed to be proportional to the ordinal number of the tree-level at which they were computed, i.e. the surprises elicited at higher tree levels are considered to be more important as those at lower levels.

In order to find the tone sequence $\mathbf{M}(t)$ of the length T which offers sufficiently great value S of the function (1) we shall use the following simple heuristic optimization algorithm:

The tth term $\mathbf{x}(t)$ is added to the already composed sequence $\mathbf{M}(t-1) = \mathbf{x}(1), \mathbf{x}(2), \ldots, \mathbf{x}(t-1)$ if the sequence $\mathbf{x}(1), \mathbf{x}(2), \ldots, \mathbf{x}(t), \mathbf{x}(t+1), \ldots, \mathbf{x}(t+s-1)$ reached the greater value S as the other sequences $\mathbf{M}(t+s-1)$ gained from the sequence $\mathbf{M}(t-1)$ by addition of all possible trial sequences $\mathbf{x}(t), \mathbf{x}(t+1), \ldots, \mathbf{x}(t+s-1)$ of the length s. The number s will be called the depth of the search. At time moments t for which it holds $T - t + 1 < s$ the depth of search is decreasing proportionally to time, i.e. $s = T - t + 1$.

All sequences $\mathbf{M}(t+s-1)$ being composed, have uniformly repeating metrorhythmics $3,1,2,1,3,1,2,1,\ldots$ defined beforehand.

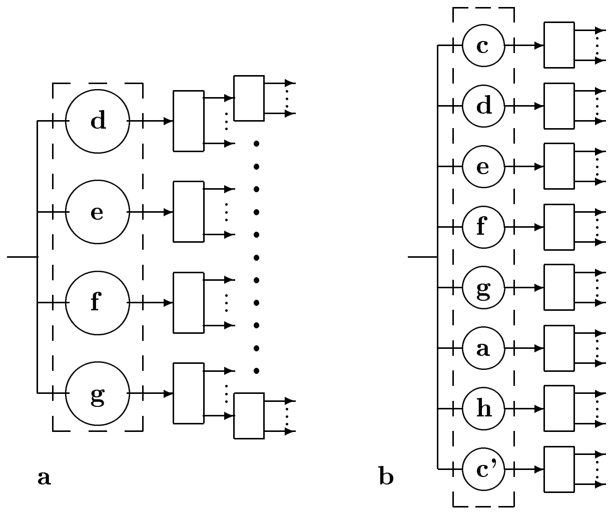

Fig. 2. *Schema of prediction trees by means of which the pitch compositional process has been performed. (a) An immature tree. (b) A mature tree. The dashed boxes denote tree nodes having the same structure as that in the tree root.*

2.1 Remark on melody completeness and closure

We suppose in accordance with the opinion of Meyer [3] that feelings of completion of a listened melody arise at a point in time when all expectations at all "architectonic levels of the perceived melody" are destroyed. In the terminology of the presented formalism, one goal of a melody termination is to inhibit the activity of all prediction trees. The above-described composing algorithm provides for that the level of surprise is increasing at the end of the composed melody since the depth of the search is decreasing during final time-steps of the compositional process. If the elicited surprise is greater than the surprise threshold of the trees, the prediction process in the trees is stopped. However, the complete solution of this problem exceeds the possibilities of the formalism presented here.

3 Computer simulation of a melody compositional process

As it has been mentioned in Sect. I.2.4., a prediction tree or a hierarchy of prediction trees may perform only one task: to evaluate the level of surprise of the pitch motions taking place in the input tone sequence from the viewpoint of the pitch sequences currently memorized in these trees. Now, a principal question appears. What pitch sequences ought to be represented and memorized in the trees so that the trees would perform the melody analysis at least partially by the manner as it is done by man ? We do not know the right answer. However, there are two extreme possible ways to arrange prediction trees. First, we may create inicialized or educated *immature trees* (see the terminology introduced in I.2.4.). Second, we can set up inicialized or educated *mature trees* . In the first case, each tree path of a prediction tree is able to memorize an arbitrary pitch sequence the terms of which are consonant with the corresponding terms of the sequence represented in this path. It means that immature trees have a great potency to memorize new pitch sequences. However, a mature tree is not able to memorize a new pitch sequence. There is a simple reason why we shall try to compose melodies both with the inicialized trees and with the educated trees. Though the tone sequence $\mathbf{M}(t)$ composed at time t represents the prediction tree education for the composing of its continuation, there is no assurance that this "education" is appropriate since the depth of search used for the composing of $\mathbf{M}(t)$ need not be sufficiently large. This problem resembles that of finding the right opening of a chess play. As the input tone sequence for education of inicialized trees we shall use three repetitions of the melody depicted in Fig. 1.

All tree arrangements proposed in what follows will consist of two immature or mature prediction trees with the same organization which will process, as it was already mentioned, the same input tone sequences with uniform metrorhythmics 3121. The first prediction tree will be started by each occurence of a tone with a strong accent ($m(i) = 3$) and a new pitch and shifted by an arbitrary accented tone ($m(i) \geq 2$) with a new pitch, while the second tree will be started by each occurence of an accented tone ($m(i) \geq 2$) with a new pitch and shifted by each onset of a tone ($m(i) \geq 1$) with a new pitch. The couples (w_1, w_2) of the tree relative importances will be set at $(0, 1)$ or at $(4, 1)$ to support the analysis of pitch motions at the hierarchy level 1 or 2, respectively. The value of the quality of discrimination of all D-units of all trees is chosen to be 1.0. For immature trees we set the D-unit threshold and the tree surprise threshold at the value 0.39 and 0.80, for mature trees at 0.90 and 1.00, respectively. The adaptivity A_o of all the SE-units is set at values $0.2, 0.4$ or 0.8.

In both immature trees all possible pitch sequences p_1, p_2, p_3 ($p_i \in \{d, e, f, g\}$) will be represented. It means that the immature tree will consits of $4^3 = 64$ pathes of the length 3 (see Fig.2a). Since just the tone doubles d-a; e-h; f-c' and g-c are consonant at the adjusted values of the D-unit threshold and discrimination quality, each pitch sequence of the length 3 can be memorized in the immature tree. In both mature trees all possible doubles p_1, p_2 ($p_i \in \{c, d, e, f, g, a, h, c'\}$) will be represented and memorized, i.e. each mature tree will consist of $8^2 = 64$ pathes of the length 2 (see Fig. 2b). The neural unit parameters chosen by individual computer simulations of both mentioned tree arrangements are shown in Table I and II together with the simulation results. The length of all the composed tone sequences is in all cases equal 32, the depth of search used in the compositional procedure is 4.

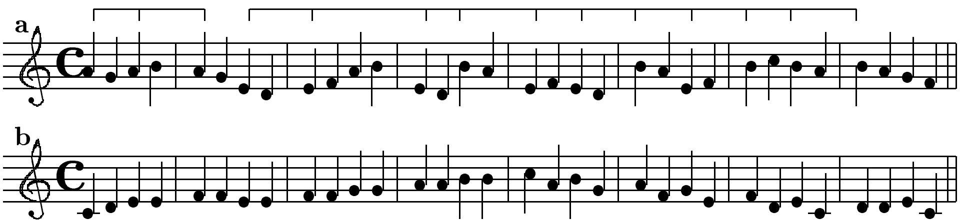

Fig. 3. *Two sequences composed by means of immature trees. (a) Sequence No. 9 of Table I. There are depicted above this notation the shifting tones which appropriately switch memory states of the tree-root SE-units. (b) Sequence No. 6 of Table I.*

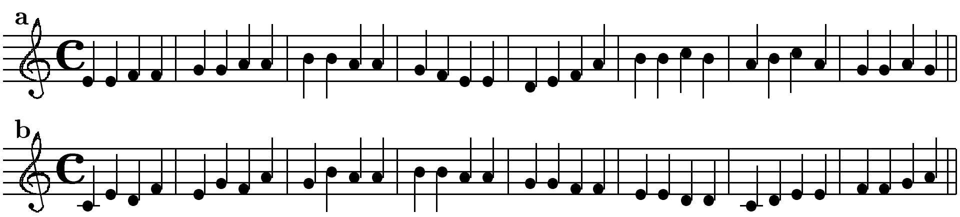

Fig. 4. *Two sequences composed by means of mature trees. (a) Sequence No. 8 and (b) sequence No. 4, 5 or 6 of Table II.*

3.1 Short discussion of the "musical quality" of the composed tone sequences

The reader is asked to pass judgment on the musical quality of the composed tone sequences by himself or herself. However, he or she ought to realize that (i) the metrorhythmis of the sequences is uniform, not optimized, (ii) they were composed by means of simple pitch prediction trees with a small prediction depth and insufficient education, c) the depth of search used in the compositional algorithm need not be sufficiently large: for the prediction trees performing the pitch motion analysis at the hierarchical level 2, it is only 2, d) the composed sequences are atonal and have not a true closure since no mechanism for musical scale detection is included in the presented model.

In spite of the mentioned facts we want to show that the composed sequences meet the basic requirements put on them. They are sufficienly smooth, single tones of them elicit surprise and make it possible for following tones to elicit surprise too, e.g. the starting tones tune appropriately the memory states of root SE-units of immature trees. It is best visible looking at the composed tone sequence which is brought in Fig. 3. The values of mean surprise are sufficiently high which gives evidence that the tones "penetrated" deeply into prediction trees.

The composed sequences sound relatively well (see e.g. those given in musical notation in Fig. 3 and Fig. 4a,b). It seems that the sequences composed with immature trees are on average more "melodic" then those with mature trees.

We were note able to find a natural melody or "artificial" tone sequence eliciting in the above defined prediction tree arrangements a greater mean level of surprise than the sequences given in Tables I and II. For example, the tone sequence depicted in Fig. 1b which is the shortest sequence with uniform metrorhythmics 3121 comprising the same pitch motions at the hierarchy level 1 and 2 as the folk melody in Fig.1 a, elicits smaller mean surprises than the corresponding tone sequences shown in Tables I and II. An analysis of this melody performed by immature trees with the same parameters as those used by composing sequences No. 3 and No. 6 of Table I, brings the mean surprise only 1.54 and 4.57, respectively.

Repetitions of two tones occurring in tone sequences composed by means of mature trees (see e.g. the sequence No. 4 of Table II) are artefacts, or boundary effects, which result from the simple fact that these trees have only three SE-units in each tree-path but they are started by each second or forth tone. "True repetitions", being frequently found in natural melodies, were obtained when we had decreased the value of surprise threshold, e.g. at 0.5. Using the same other tree parameters as in the case of composing the sequence No. 7 of Table I, we got the sequence efed eded edef eded efed edef eded edef. This sequence comprises monotonal repetitions being composed by means of only one tree path. The low tree surprise threshold did not make it possible to tune the SE-unit in tree-root in such manner that another tree-path could be opened. Maybe, if we used the composing procedure with a greater depth of search then 4, we would obtain another result.

4 General conclusions

The presented neural model for the processing of temporal tone sequences is able to explain the pitch processing invariancy in regard to the sequence tempo and tone durations but it cannot account for the pitch sequence processing invariancy in regard to a constant frequency shift of all the sequence tones. We suppose that for explanation of the last-mentioned phenomenon it is necessary to build up a motor theory of music perception in the same manner as it was done in case of speech perception by Liberman and his coworkers (for the recent development of this theory see Ref. 7). Although the presented model does not represent a motor theory of music perception, it incorporates some features of such a theory. For example, the chosen definition of tree starting and shifting tones comes out from the assumption that the sequence of motor commands which control the movement of vocal chords during brain processing of a sequence of tones (chords) determines the perceived pitch motion. If we suppose further that the vocal chords "reproducing" the perceived pitches are controlled by two separate channels, one for the slow tonic commands and the other for the fast phasic commands, we might elucidate the fact that the piece of a melody which starts and ends on the tonic of a musical scale, is experienced by us as a melodic perception unit. In this case, the origin of this percept and its invariancy regarding to the constant frequency shift, can be explained by the assumption that the brain state corresponding to this percept is evoked by the choice of that set of phasic commands which, imposed on an appropriate tonus motor command, control vocal chords during reproduction of the perceived melody.[8] These considerations are supported by the opinion of the music theoreticians [5,9] treating a pitch as a bidimensional attribute. Dowling [9] has stated that "a pitch would be categorized in terms of two dimensions: one for pitch level within the octave and the other for octave level". Today, such a bidimensional process for the pitch estimation could be modelled by two subsequent relaxation processes occuring in the modified Hopfield spin models of neural nets for invariant pattern reognition proposed recently by Dotsenko [10] and by Kree and Zippelius [11]. It will be the subject of one of our next papers. Here we have composed only single-lined sequences comprising tones from a single diatonic musical scale.

This paper ends the project MACSIM (Mathematical Analyser and Composer of SImple Melodies) which was started by Fedor's paper.[8]

Table I. Tone sequences composed by means of the prediction tree arrangement consisting of two immature prediction tree with different starting and shifting tones (see text). Parameter w_1 and w_2 represents the importance weight of the first and second tree, respectively. A_o is the adaptivity constant of SE-units, S is the mean level of surprise elicited by the corresponding sequence. There is marked in the column "Education" whether prediction trees before running compositional procedure were educated or not.

No	w_1	w_2	A	Education	Composed Sequence								S
1	0	1	0.2	no	cdef	gfed	efed	cdcf	gac'h	ffga	gagf	cded	3.11
2	0	1	0.4	no	cdef	gfed	efed	cdgf	gfef	gaha	gfga	hc'hc'	3.31
3	0	1	0.8	no	cdef	gfed	efed	cdgf	gfef	gahg	gfga	gahc'	3.35
4	4	1	0.2	no	cdee	fdee	ffgg	aagh	aghh	ac'ha	c'c'hh	c'c'ha	8.83
5	4	1	0.4	no	cdee	fdee	ffgg	aagh	aghh	c'ahh	aagg	ffgf	8.61
6	4	1	0.8	no	cdee	ffee	ffgg	aahh	c'ahg	afge	fdec	ddec	7.69
7	0	1	0.2	yes	agah	aged	efha	hc'ed	edef	hc'ha	haef	efgf	2.76
8	0	1	0.4	yes	agah	aged	efah	efed	haed	efhc'	hahc'	efed	2.71
9	0	1	0.8	yes	agah	aged	efah	edha	efed	haef	hc'ha	hagf	2.73
10	4	1	0.2	yes	afed	ffge	ffgh	aagh	ac'hc'	ac'ha	dcdf	eefc	7.20
11	4	1	0.4	yes	afed	ffge	ffgh	aagh	ac'hc'	afha	dcdf	eefc	7.16
12	4	1	0.8	yes	afed	ffge	fdfd	eefd	echf	aagg	aahg	afgc	6.50

Table II. Tone sequences composed by means of the prediction tree arrangement consisting of two mature prediction tree with different starting and shifting tones (see text). The meaning of the tree arrangement parameters are the same as in Table I.

No	w_1	w_2	A	Education	Composed Sequence								S
1	0	1	0.2	no	ccdd	eeff	ggaa	hhc'h	aagg	ffee	dcda	hhah	2.24
2	0	1	0.4	no	ccdd	eeff	ggaa	hhaa	ggff	eedc	dhc'c'	hc'ha	2.24
3	0	1	0.8	no	ccdd	eeff	ggaa	hhaa	ggff	edce	ddcd	cefg	2.26
4	4	1	0.2	no	cedf	egfa	ghaa	hhaa	ggff	eedd	cdee	ffga	6.45
5	4	1	0.4	no	cedf	egfa	ghaa	hhaa	ggff	eedd	cdee	ffga	6.49
6	4	1	0.8	no	cedf	egfa	ghaa	hhaa	ggff	eedd	cdee	ffga	6.52
7	0	1	0.2	yes	aahh	aagg	ffgg	aahh	c'haf	efge	fagf	egag	2.17
8	0	1	0.4	yes	eeff	ggaa	hhaa	gfee	defa	hhc'h	ahc'a	ggag	2.20
9	0	1	0.8	yes	ffgg	aahh	aagf	ecdd	efgh	c'haa	hc'aa	gagf	2.19
10	4	1	0.2	yes	ecde	cedf	egfa	ghaa	hhaa	ggff	efge	fded	6.35
11	4	1	0.4	yes	ecde	cedf	egfa	ghaa	hhaa	ggff	efge	fded	6.43
12	4	1	0.8	yes	ecde	cedf	egfa	ghaa	hhaa	ggff	efge	fded	6.49

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